# TRANSIENT DISPLACEMENT FIELDS IN HEXAGONAL CRYSTALS AND TRANSVERSALLY ISOTROPIC MEDIA

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The structure of the disturbed region and the geometry of the wave front is investigated under the condition that a concentrated source of the instantaneous-pulse type is acting in an unbounded transversally isotropic medium. The regions of permissible values of the anisotropy coefficient introduced in [1] for transversally isotropic media on the basis of conditions of the elastic energy's positive-definiteness and hyperbolicity conditions are determined. It is suggested that motion of the medium occurs under conditions of plane deformation.

§1. A detailed investigation has been conducted in [1] of the characteristics of the wave fields in elastic anisotropic media for the case of concentrated sources of pulse-type disturbances. The geometry of the wave front, the roots of the characteristic equation, and some physical characteristics of the medium have been investigated. In order to describe the properties of specific media, the anisotropy coefficient $\Delta_A = (c_1 - c_3)/c_2$ is is introduced for the case $c_1 = c_4$, where $c_1$, $c_2$, $c_3$, and $c_4$ are coefficients of the equations

$$c_1\partial^2 u/\partial x^2 + c_2\partial^2 w/\partial x\partial z + c_3\partial^2 u/\partial z^2 - \rho\partial^2 u/\partial t^2 = \rho a_1 f;$$
$$c_3\partial^2 w/\partial x^2 + c_2\partial^2 u/\partial x\partial z + c_4\partial^2 w/\partial z^2 - \rho\partial^2 w/\partial t^2 = \rho a_2 f,$$

which describe the medium's motion under the conditions of plane deformation.

Here u and w are the displacement components along the x and z axes. The direction of the coordinate axes is chosen as a function of the specific kind of elastic symmetry (as in [1]). If the source of the disturbances is localized at the origin of coordinates and is a concentrated pulse, the right-hand sides should be taken in the form

$$f = \delta(x)\delta(z)\delta(t).$$

Media possessing cubic symmetry are discussed in detail in [1]. It is shown that the entire region of permissible values of $\Delta_A$ and $\alpha (\alpha = c_3/c_1)$ is divided into four regions. The presence and arrangement of lacunae and the form of the roots of the characteristice equation are determined as a function of which of the regions a point having the coordinates $\Delta_A$ and $\alpha$ falls into. The boundaries of the permissible values of $\Delta_A$ and $\alpha$ are determined by the conditions of hyperbolicity and positive-definiteness of the elastic energy. The two parameters $\Delta_A$ and $\alpha$ completely determine the behavior of the medium under the conditions of plane deformation in the case of cubic symmetry.

Let us now consider hexagonal crystals and transversally isotropic media. V. L. German proved a theorem in 1944 which generalizes Neumann's principle to the case of complex anisotropic media [2]: "If a medium possesses an axis of symmetry of order n, then it is axially isotropic with respect to this axis for all physical properties whose characteristics are determined by tensors of rank r, if $r < n$. Thus, for example, the plane perpendicular to this axis will be a plane of isotropy for elastic properties ($r = 4$) already upon the existence of fifth-order ($n = 5$) symmetry." This theorem permits not drawing distinctions between hexagonal crystals and transversally isotropic media and has important practical application in the analysis of the symmetry of the elastic properties of multilayered media comprised of orthotropic layers. All the directions in the plane of a lamina of such a material are equivalent if the angle between the direction of the fibers in adjacent layers is less than $\varphi = 2\pi/5 = 72°$. The plane of a lamina is in this case the plane of isotropy, and the axis perpendicular to it is the axis of symmetry of infinite order. Thus, all layered materials having stellate structure are transversally isotropic if the angle between the direction of the fibers in ad-

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jacent layers is $\varphi < 72°$. German's theorem permits extending without alteration the results derived for transversally isotropic media to all media which have at each point an axis of symmetry of order $n \geq 5$, in particular, to hexagonal crystals ($n = 6$). Therefore, all the discussions are carried out for transversally isotropic media.

The expression for the elastic energy in the case of plane deformation is written in the form

$$2W = c_1\varepsilon_{xx}^2 + c_4\varepsilon_{zz}^2 + 2(c_2 - c_3)\varepsilon_{xx}\varepsilon_{zz} + c_3\varepsilon_{xz}^2,$$

where $\epsilon_{ji}$ is the deformation tensor.

The conditions of positive-definiteness W are of the form

$$c_1c_4 - (c_2 - c_3)^2 > 0$$

and agree with the condition of the realness of the roots to Rayleigh's equation [3]. Upon the condition $c_1 = c_4$, we obtain

$$c_1^2 > (c_2 - c_3)^2. \tag{1.1}$$

According to [3], the conditions (1.1) are always fulfilled if $\alpha < 1$, i.e., $c_3 < c_1$, and, consequently, they do not impose restrictions on $\Delta_A$ when $\alpha < 1$. The lower boundary of permissible values of $\Delta_A$ in the interval $\alpha < 1$ is determined by the hyperbolicity conditions and coincides with the boundary cited in [1].

Additional restrictions on $\Delta_A$, namely, $\Delta_A < \Delta_A^{\pm}(\alpha)$, $\alpha < 1$, which were derived in [1] for cubic crystals, are caused by the conditions of positive-definiteness of the elastic energy in the case of a triaxial deformation.

In the case of systems not pertaining to cubic crystals the conditions of the elastic energy's positive-definiteness upon a triaxial deformation relate the elastic constants which do enter into the expressions for the $c_i$ (whose number is always four) and which do not enter into the expressions for the $c_i$ in the case of transversally-isotropic media

$$c_1 = a_{11},\ c_2 = a_{13} + a_{44},\ c_3 = a_{44},\ c_4 = a_{33};$$

$a_{12}$ does not enter into the expressions for the $c_i$. We have for orthotropic materials

$$c_1 = a_{11},\ c_2 = a_{13} + a_{55},\ c_3 = a_{55},\ c_4 = a_{33}.$$

The five remaining elastic constants do not enter into the expressions for the $c_i$. Since the conditions of positive-definiteness linking all the elastic constants for each kind of elastic symmetry (regardless of whether plane or triaxial deformation is being considered) must always be fulfilled for actual media, it is necessary to take these conditions into account in determining the permissible values of $\Delta_A$ and $\alpha$. In the case of transversally-isotropic media the conditions of the elastic energy's positive-definiteness, which link all five elastic constants, are of the form

$$a_{44} > 0, \qquad a_{11} > |a_{12}|, \qquad (a_{11} + a_{12})a_{33} > 2a_{13}^2,$$

whence

$$c_3 > 0, \quad k_{12} < 1, \quad c_1c_4 > \frac{2}{k_1}(c_2 - c_3)^2, \tag{1.2}$$

where

$$k_{12} = |a_{12}|/c_1;\ k_1 \equiv 1 + k_{12};\ k_1 < 2.$$

In the particular case under discussion, in which $c_1 = c_4$, we obtain from (1.2) the fact that in the interval $\alpha_k < \alpha < 1$ all values of $\Delta_A$ from the interval $\Delta_{A1}(\alpha) < \Delta_A < \Delta_{A2}(\alpha)$, where $\Delta_{A1}(\alpha) = (1-\alpha)/(\alpha + \alpha_A)$, $\Delta_{A2}(\alpha) = (1-\alpha)/(\alpha - \alpha_k)$, and $\alpha_k = \sqrt{k_1/2}$, are permissible. All values $\Delta_A > \Delta_{A2}(\alpha)$ are permissible in the interval $0 < \alpha < \alpha_k$. Since $k_1 < 2$ (provided that $c_1 > |a_{12}|$), we obtain $\alpha_k < 1$. Curves of $\Delta_{A1}(\alpha)$ and $\Delta_{A2}(\alpha)$ are shown in Fig. 1 [curves 5' and 5'', respectively; curve 5 is $\Delta_A^{\pm}(\alpha)$].

The curves 1-4 differ in no way from the corresponding curves cited in [1] in Fig. 3.2.

Since $\alpha_k < 1$, the curve of $\Delta_{A1}(\alpha)$ passes above $\Delta_{A\Gamma}(\alpha)$ (curve 4), which is defined by the hyperbolicity conditions, and in this way decreases the region of permissible values of $\Delta_A$ in comparison with cubic crystals. However, the curve of $\Delta_{A2}(\alpha)$ passes to the right of the curve of $\Delta_A^{\pm}(\alpha)$, which corresponds to an increase in the region of permissible values of $\Delta_A$ and $\alpha$ in comparison with cubic crystals. The values

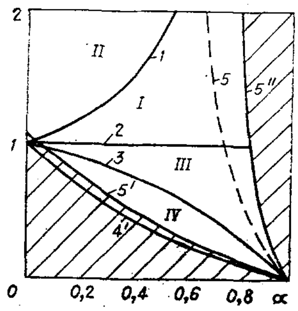


Fig. 1

$\Delta_A < \Delta_{A1}(\alpha)$ and $\Delta_A > \Delta_{A2}(\alpha)$ are forbidden (crosshatched area). In particular, all $\alpha > 1$ fall in this region. Just as in the case of cubic crystals, only the values $c_3 < c_1$ are permissible, i.e., the velocity of quasitransverse waves are always less than the velocity of quasilongitudinal waves.

One of the important consequences which follows from the cited results is the fact that the boundaries of permissible values of $\Delta_A$ and $\alpha$ are determined not by the conditions of positive-definiteness W for plane deformation, which are of the identical form for materials of different symmetry, but from the conditions of positive-definiteness in the case of triaxial deformation, in which W depends on the complete set of elastic constants, whose number is different for different classes of elastic symmetry and is greater than three in all cases except cubic crystals. In particular, the boundaries of permissible values are fixed only for cubic crystals. In all other cases their position depends on the elastic constants which do not enter into the $c_i$.

<u>THEOREM OF CONNECTIVITY.</u> The transformation

$$c_1' = c_1 + c_2 + c_3, \quad c_3' = c_1 + c_3 - c_2, \quad c_2' = c_1 - c_3, \tag{1.3}$$

converts media of region I to media of region III and media of region II to media of region IV (see Fig. 1). The inverse transformation results in the reverse conversion.

<u>Proof.</u> It is sufficient for proving this theorem to show that $\Delta_A' = (c_1' - c_3')/c_2' < 1$ if $\Delta_A = (c_1 - c_3)/c_2 > 1$; the permissible values of $\Delta_A$ are converted to permissible $\Delta_A'$; forbidden $\Delta_A$ – to forbidden $\Delta_A'$; the boundary $\Delta_{A*}(\alpha)$ is converted to the boundary $\Delta_A^0(\alpha')$, and the boundary of region I and III, i.e., the straight line, $\Delta_A = 1$, is not altered. The fact that the boundary $\Delta_{A*}(\alpha)$ separating regions I and II is converted to the boundary $\Delta_A^0(\alpha')$ separating regions III and IV is shown in [1]. From (1.3) we obtain

$$c_1'/c_1 = \left[(1 + \alpha) + \sqrt{1 + \alpha^2 - \gamma}\right]/2,$$

$$c_3'/c_1 = \left[(1 + \alpha) - \sqrt{1 + \alpha^2 - \gamma}\right]/2, \quad c_2'/c_1 = 1 - \alpha,$$

consequently,

$$\Delta_A' = \frac{c_1' - c_3'}{c_2'} = \frac{\sqrt{1 + \alpha^2 - \gamma}}{1 - \alpha} = \frac{1}{\Delta_A}.$$

Thus, if $\Delta_A > 1$, then $\Delta_A' < 1$, and we have $\Delta_A' > 1$ when $\Delta_A < 1$. From this it follows in particular that $\Delta_A = 1$ is converted into $\Delta_A' = 1$. The transformation (1.3) does not alter the values of the elastic potential, since it reduces to a simple rotation of the coordinate axes by an angle of 45°. It was shown above that both in the case of cubic crystals and in the case of transversally isotropic media the boundaries of permissible values of $\Delta_A$ are completely determined in the case of a triaxial deformation by just the same conditions of the elastic potential's positive-definiteness. We obtain the fact that the transformation (1.3) cannot be derived beyond the boundary of permissible values of $\Delta_A$. The theorem is proven. It follows from this theorem that: 1) all media with $c_1 = c_4$ consist of media of the first group and media obtained from them by a rotation of the coordinate axes by 45°; 2) the transformation (1.3) converts roots of a characteristic equation of the first type to roots of a characteristic equation of the second and third types (according to the classification of [4]); and 3) if the coefficients of the equations for two media are connected by the relations (1.3), the direction of a concentrated pulse load acting in the first medium ($\Delta_A > 1$) makes an angle of 45° with the direction of the force acting in the second medium ($\Delta_A < 1$), they are equal in magnitude, and the wave patterns in both media coincide completely in the case of a rotation of the media by 45° with respect to each other (the displacements are identical at the same points upon the congruence of the wave patterns). Hence, one can draw the following conclusions.

1. It is not at all obligatory to have recourse to the solution of the equation $t - \theta_n x - \mu_n(\theta_n) z = 0$ for

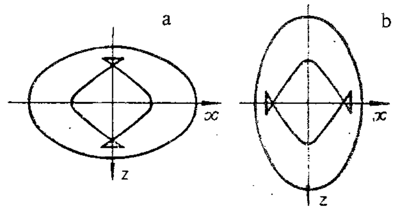


Fig. 2

calculating the displacement curves at points on lines which cross at an angle of 45° or 135° with respect to the coordinate axes [6].

It is sufficient to replace the values of the coefficients $c_1$, $c_2$, and $c_3$ according to Eqs. (1.3) in the expressions (3.4) of [6], which give in explicit form the solutions for points lying on the coordinate axes.

2. It is sufficient to know the type of roots of the equations $\theta_n$ and an algorithm for their calculation for media only of the first group, which is cited in [7], in order to calculate the displacement fields in the case of a concentrated pulse source for any media with $c_1 = c_4$.

In all other cases except cubic crystals the permissible range of values of $\alpha$ when $\Delta_A = 1$ is greater than for isotropic media, since the possibility occurs of satisfying the conditions of positive-definiteness when $\alpha > 0.75$ owing to the variation of the elastic constants which do not enter into $c_1$, $c_2$, $c_3$, and $c_4$.

§2. One can extend the cited results to media with $c_1 \neq c_4$ which is more typical of transversally isotropic media. Thus, cubic crystals are not considered, since $c_1 = c_4$ always for them. In this case it is necessary to take $\Delta_A$ in the form

$$\Delta_A = (\sqrt{c_1 c_4} - c_3)/c_2 = (1 - \sqrt{\alpha\beta})/\sqrt{1 + \alpha\beta - \gamma},$$

where

$$\alpha = c_3/c_1; \quad \beta = c_3/c_4; \quad \gamma = 1 + \alpha\beta - c_2^2/c_1 c_4.$$

Upon such a definition $\Delta_A \geq 1$ corresponds, as before, to media of the first group.

The principal difference among media consists of the fact that configurations of wave fronts having two lacunae are possible for a concentrated pulse source in an unbounded medium.

Under the conditions

$$\alpha < \beta, \ \alpha(\beta + 1) < \gamma < \beta(\alpha + 1) \tag{2.1}$$

lacunae lie on the z axis (Fig. 2a). Under the conditions

$$\alpha > \beta, \ \beta(\alpha + 1) < \gamma < \alpha(\beta + 1) \tag{2.2}$$

the lacunae are located on the x axis (Fig. 2b).

The set of parameters $\alpha$, $\beta$, and $\Delta_A$ already determines the point in space, and the boundaries separating the regions are surfaces. The cross section of these surfaces with the plane $\alpha = \beta$ determines the curves 1-4, 5', and 5'' (see Fig. 1). The values of $\Delta_A$ are restricted from below by the surface

$$\Delta_{A\Gamma}(\alpha, \beta) = (1 - \sqrt{\alpha\beta})/(1 + \sqrt{\alpha\beta}).$$

The hyperbolicity conditions are not fulfilled below this boundary. The conditions (2.1) and (2.2) determine regions of which there are none in the case $\alpha = \beta$.

The line of intersection of the surface $\Delta_A = \Delta_{A\Gamma}(\alpha, \beta)$ with the plane $\Delta_A = 0$ is the hyperbola $\alpha\beta = 1$. As $\alpha\beta \to \infty$, the region of the values of $\Delta_A$ permitted by the hyperbolicity conditions approaches $-1$, $-1 < \Delta_A < \infty$, just as when $\alpha = \beta$. The straight lines $\alpha = 0$, $\Delta_A = 1$ and $\beta = 0$, $\Delta_A = 1$, respectively, are the intersection lines of the surface $\Delta_{A\Gamma}(\alpha, \beta)$ with the coordinate planes $\alpha = 0$ and $\beta = 0$.

Media of the first group [1, 3] occupy the region in which both the conditions

$$[2\beta(1 + \alpha) - \gamma(1 + \beta)] \geqslant -|\beta - 1|\sqrt{\gamma^2 - 4\alpha\beta};$$
$$[2\alpha(1 + \beta) - \gamma(1 + \alpha)] \geqslant -|\alpha - 1|\sqrt{\gamma^2 - 4\alpha\beta},$$

are fulfilled, inside a cylindrical region with the rectangular base $\Delta_A = 1$, $\alpha < 1$, $\beta < 1$ and lateral boundaries parallel to the coordinate planes. Points arranged between the surfaces $\Delta_{A\alpha}(\alpha, \beta)$ and $\Delta_{A\beta}(\alpha, \beta)$, i.e.,

$$\Delta_{A\alpha}(\alpha, \beta) < \Delta_A < \Delta_{A\beta}(\alpha, \beta), \ \alpha < \beta,$$

where

$$\Delta_{A\alpha}(\alpha, \beta) = (1 - \sqrt{\alpha\beta})/\sqrt{|1-\alpha|};$$

$$\Delta_{A\beta}(\alpha, \beta) = (1 - \sqrt{\alpha\beta})/\sqrt{|1-\beta|},$$

correspond to media which satisfy the conditions (2.1). A region also situated between the given surfaces but located on the other side of the plane $\alpha = \beta$, i.e, where $\beta < \alpha$, corresponds to the media (2.2). The surface $\Delta_{A\alpha}(\alpha, \beta)$ already lies above the surface $\Delta_{A\beta}(\alpha, \beta)$. Regions in which both kinds of media are realized disappear as $\alpha \to \beta$, since the surfaces $\Delta_{A\alpha}$ and $\Delta_{A\beta}$ intersect when $\alpha = \beta$. The line of their intersection defines the curve $\Delta_A^0 = \sqrt{1-\alpha}$ in the $\Delta_A$, $\alpha$ plane (see curve 3 in Fig. 1). Media having four lacunae lying on the x and z axes correspond to the points of the region which lie below $\Delta_{A\alpha}(\alpha, \beta)$ when $\alpha < \beta$ and below $\Delta_{A\beta}(\alpha, \beta)$ when $\beta < \alpha$ [4]. This region is limited from below by the surface $\Delta_{A1}(\alpha, \beta)$. The region $\alpha\beta > 1$ and $\Delta_A > 1$, which is limited by the cylindrical surface $\alpha\beta = 1$ and by the plane $\Delta_A = 1$ from below, is forbidden, since the conditions of the elastic potential's positive-definiteness are not fulfilled there even for plane deformation (moreover, media pertaining to this region have imaginary Rayleigh wave velocities [3]).

The conditions of positive definiteness W in case of a triaxial deformation are not fulfilled in the regions above the surface

$$\Delta_{A2}(\alpha, \beta) = (1 - \sqrt{\alpha\beta})/(\sqrt{\alpha\beta} - \alpha_k)$$

and below the surface

$$\Delta_{A1}(\alpha, \beta) = (1 - \sqrt{\alpha\beta})/(\alpha_k + \sqrt{\alpha\beta})$$

and these regions are forbidden. In particular, this includes all the values of $\alpha$ and $\beta$ where $\alpha\beta > 1$, i.e., $c_3 > \sqrt{c_1 c_4}$; however $\alpha > 1$ and $\beta > 1$ are permissible separately but such that $\alpha\beta < 1$ always ($\alpha > 1$ and $\beta > 1$ are not simultaneously permissible). The position of the surfaces $\Delta_{A1}$ and $\Delta_{A2}$ in space depends on the quantity $\alpha_k$ i.e., finally on the elastic constant $a_{12}$ (or the Poisson coefficient for the y axis in the case of extension along the x axis, which is equal to $k_{12}$).

§3. Young's modulus for transversally isotropic media is defined by the expression

$$1/E = S_{11} \sin^4\theta + S_{33} \cos^4\theta + (S_{44} + 2S_{13}) \sin^2\theta \cos^2\theta, \tag{3.1}$$

where $\theta$ is the angle figured from the main axis;

$$S_{33} = (a_{11} + a_{12})S, \ S_{44} = 1/a_{44}, \ (S_{11} + S_{12}) = a_{33}S,$$

$$(a_{11} - a_{12})(S_{11} - S_{12}) = 1, \quad S_{13} = -a_{13}S; \quad S = S_{33}(S_{11} + S_{12}) - 2S_{13}^2.$$

The conditions of positive-definiteness for the moduli of elasticity $S_{ij}$ are of the same form as for the elastic constants $a_{ij}$,

$$S_{44} > 0, \quad S_{11} > |S_{12}|, \quad S_{33}(S_{11} + S_{12}) > 2S_{13}^2. \tag{3.2}$$

Substituting the second of the inequalities (3.2) into the third, we obtain

$$S_{33}S_{11} > S_{13}^2, \quad |S_{13}| < \sqrt{S_{11}S_{33}}. \tag{3.3}$$

Since $S_{11}$, $S_{33}$, and $S_{44}$ are all positive, $E > 0$ always when $S_{13} > 0$. When $S_{13} < 0$, we write Eq. (3.1) in the form

$$1/E = P(\theta) + S_{44} \sin^2\theta \cos^2\theta,$$
$$P/(\theta) \equiv S_{11} \sin^4\theta + S_{33} \cos^4\theta - 2\,|S_{13}| \sin^2\theta \cos^2\theta. \tag{3.4}$$

Substituting (3.3) into (3.4), we find

$$P_1(\theta) \equiv S_{11} \sin^4\theta + S_{33} \cos^4\theta - 2\sqrt{S_{11}S_{33}} \sin^2\theta \cos^2\theta = (\sqrt{S_{11}} \sin^2\theta - \sqrt{S_{33}} \cos^2\theta)^2 > 0,$$

consequently, $P(\theta) > 0$.

Thus, Young's modulus (3.1) is always positive, just as for cubic crystals. In the case of cubic crystals the anisotropy factor [5] $\xi_A = 2a_{44}/(a_{11} - a_{12})$, which characterizes the anisotropy of shears, is used to characterize the elastic properties. The isotropy condition $\xi_A = 1$ represents the equality of two wave velocities in the direction ⟨110⟩ out of the existing three, which are written in the case of cubic crystals in the form

$$c_l = \sqrt{(a_{11} + a_{12} + 2a_{44})/2\rho};$$
$$c_{t1} = \sqrt{(a_{11} - a_{12})/2\rho}; \; c_{t2} = \sqrt{a_{44}/\rho}.$$

Thus, the coefficient $\xi_A$ characterizes the ratio of the velocities $c_{t1}$ and $c_{t2}$

$$\xi_A = (c_{t2}/c_{t1})^2.$$

The velocities $c_{t1}$ and $c_{t2}$ are equal to each other on the condition that

$$2a_{44} = a_{11} - a_{12}. \quad (3.5)$$

Assuming, according to [1], $a_{11} = c_1$, $a_{12} + a_{44} = c_2$, and $a_{44} = c_3$, we obtain $c_2 = c_1 - c_3$ or $\Delta_A = 1$ from (3.5), i.e., $\Delta_A = 1$ follows from the relationship $\xi_A = 1$, and vice versa. Thus cubic crystals with $\Delta_A = 1$ are actually isotropic media, and the coefficient $\Delta_A$ is completely equivalent to $\xi_A$ for cubic crystals.

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